IDR/QR: An Incremental Dimension Reduction Algorithm via QR Decomposition

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Abstract—Dimension reduction is a critical data preprocessing step for many database and data mining applications, such as efficient storage and retrieval of high-dimensional data. In the literature, a well-known dimension reduction algorithm is Linear Discriminant Analysis (LDA). The common aspect of previously proposed LDA-based algorithms is the use of Singular Value Decomposition (SVD). Due to the difficulty of designing an incremental solution for the eigenvalue problem on the product of scatter matrices in LDA, there has been little work on designing incremental LDA algorithms that can efficiently incorporate new data items as they become available. In this paper, we propose an LDA-based incremental dimension reduction algorithm, called IDR/QR, which applies QR Decomposition rather than SVD. Unlike other LDA-based algorithms, this algorithm does not require the whole data matrix in main memory. This is desirable for large data sets. More importantly, with the insertion of new data items, the IDR/QR algorithm can constrain the computational cost by applying efficient QR-updating techniques. Finally, we evaluate the effectiveness of the IDR/QR algorithm in terms of classification error rate on the reduced dimensional space. Our experiments on several real-world data sets reveal that the classification error rate achieved by the IDR/QR algorithm is very close to the best possible one achieved by other LDA-based algorithms. However, the IDR/QR algorithm has much less computational cost, especially when new data items are inserted dynamically.

Index Terms—Dimension reduction, linear discriminant analysis, incremental learning, QR Decomposition, Singular Value Decomposition (SVD).

1 Introduction

The problem of dimension reduction has recently received broad attention in areas such as databases, data mining, machine learning, and information retrieval [3], [5], [10], [11], [24]. Efficient storage and retrieval of high-dimensional data is one of the central issues in database and data mining research. In the literature, many efforts have been made to design multidimensional index structures [4], such as *R*-trees, R^*-trees, *X*-trees, SR-tree, etc., for speeding up query processing. However, the effectiveness of queries using any indexing scheme deteriorates rapidly as the dimension increases, which is the so-called *curse of dimensionality*. A standard approach to overcome this problem is dimension reduction, which transforms the original high-dimensional data into a lower-dimensional space with limited loss of information. Once the high-dimensional data is mapped into a low-dimensional space, indexing techniques can be effectively applied to organize this low-dimensional space and facilitate efficient retrieval of data [24]. A further advantage of such dimension reduction is that it can improve data quality through the removal of noise [1]. Thus, dimension reduction is an important data preparation step for many data mining and database applications.

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The goal of dimension reduction can be either feature transformation, which aims to find a linear combination of the original features, or feature selection, which selects a subset of features from the original features. The setting can be unsupervised or supervised, depending on the availability of the class label. In this paper, we focus on supervised dimension reduction by applying feature transformation.

Linear Discriminant Analysis (LDA) is a well-known algorithm for supervised dimension reduction [12], [17], [21]. LDA computes a linear transformation by maximizing the ratio of between-class distance to within-class distance, thereby achieving maximal discrimination. A key problem with LDA is that the scatter matrices used for between-class and within-class distances can sometimes become singular. In the past, many LDA extensions have been developed to deal with this singularity problem. There are three major extensions: regularized LDA, PCA+LDA, and LDA/GSVD. The common aspect of these algorithms is the use of the Singular Value Decomposition (SVD) or Generalized Singular Value Decomposition (GSVD). The difference among these LDA extensions is as follows: Regularized LDA [16] increases the magnitude of the diagonal elements of the scatter matrix by adding a scaled identity matrix; PCA+LDA [2] first applies Principal Component Analysis (PCA) on the raw data to get a more compact representation so that the singularity of the scatter matrices is decreased; LDA/GSVD [22], [36] solves a trace optimization problem using GSVD.

The above LDA extensions have certain limitations. First, SVD or GSVD requires that the whole data matrix be stored in main memory. This requirement makes it difficult for these LDA extensions to scale to large data sets. Also, the computational cost of SVD or GSVD on large data matrices is very high and can significantly degrade the performance of these algorithms when dealing with large data sets. Finally, in many practical applications, an acquisition of a representative training data is expensive and time-consuming. It is thus common to have a small chunk of data available over a period of time. In such settings, it is necessary to develop an algorithm that can run in an incremental fashion to accommodate the new data. However, since it is difficult to design an incremental solution of the eigenvalue problem on the product of scatter matrices of large size, little effort has been made to design LDA-like algorithms that can be updated incrementally to incorporate new data items as they become available.

The goal of this paper is to design an efficient and incremental dimension reduction algorithm while preserving competitive classification performance. More precisely, when we perform classification on the reduced dimensional data generated by the proposed algorithm, the achieved classification accuracy should be comparable to the best possible classification accuracy achieved by other LDA-based algorithms.

In this paper, we design an LDA-based, incremental dimension reduction algorithm, called IDR/QR, which applies QR Decomposition rather than SVD or GSVD. The algorithm has two stages: The first stage maximizes the separability between different classes. This is accomplished by QR Decomposition. The distinct property of this stage is its low time and space complexity. The second stage incorporates both between-class and within-class information by applying LDA on the "reduced" scatter matrices resulting from the first stage. Unlike other LDA-based algorithms, IDR/QR does not require that the whole data matrix be in main memory, which allows our algorithm to scale to very large data sets. Also, our theoretical analysis indicates that the computational complexity of IDR/QR is linear in the number of the data items in the training set as well as the number of dimensions. More importantly, the IDR/QR algorithm can work incrementally. When new data items are inserted dynamically, the computational cost of the IDR/QR algorithm can be kept low by applying efficient QR-updating techniques.

Finally, we have conducted extensive experiments on several well-known real-world data sets. The experimental results show that the IDR/QR algorithm can be an order of magnitude faster than SVD or GSVD-based LDA algorithms, and that the classification error rate of IDR/QR is very close to the best possible one achieved by other LDA-based algorithms. Also, in the presence of dynamic updating, IDR/QR can be an order of magnitude faster than SVD or GSVD-based LDA algorithms, while still achieving comparable accuracy.

Overview: The rest of the paper is organized as follows: Section 2 introduces related work. In Section 3, we review LDA. A batch implementation of the IDR/QR algorithm is presented in Section 4. Section 5 describes the incremental implementation of the IDR/QR algorithm. A comprehensive empirical study of the performance of the proposed algorithms is presented in Section 6. We conclude in Section 7 with a discussion of future work.

2 Related Work

Principal Component Analysis (PCA) is one of the standard and well-known methods for dimension reduction [23]. PCA transforms a number of (possibly) correlated variables into a (smaller) number of uncorrelated variables called *principal components*. The basic idea in PCA is that the first few principal components account for most variances. Because of its simplicity and ability to extract the highly global structure of the whole data set, PCA is widely used in computer vision [35]. Linear Discriminant Analysis (LDA) is another well-known algorithm for dimension reduction. LDA transforms the original data to a low-dimensional space by maximizing the ratio of between-class distance to within-class distance. It has been applied to various domains including text retrieval [6], face recognition [2], [28], [34], and microarray data classification [13]. A comparative study of PCA and LDA can be found in [2], [34]. Previous studies show that LDA is competitive with PCA in classification. The rationale behind this lies in the fact that LDA makes use of the class label information, while PCA is unsupervised. Note that when the class label is not available, LDA is not applicable.

Most previous work on PCA and LDA require that all the training data be available before the dimension reduction step. This is known as the *batch method*. There is some recent work in vision and numerical linear algebra literature for computing PCA incrementally [7], [19]. Despite the popularity of LDA in the vision community, there is little work for computing it incrementally. The main difficulty is the involvement of the eigenvalue problem of the product of scatter matrices, which is hard to maintain incrementally. Although iterative algorithms have been proposed for neural network-based LDA [8], [27], they require $O(d^2)$ time for one step updating, where d is the dimension of the data. An improved algorithm was proposed in [15]; it is mentioned there that the update time of the algorithm is about half that of the previous algorithms. This is, however, still $O(d^2)$, which can be expensive, when the data has high dimension.

Maximum Margin Criterion (MMC) was recently proposed in [26] for dimension reduction. The optimal transformation is computed by maximizing the sum of all interclass distances. MMC does not involve the inversion of scatter matrices and, thus, avoids the singularity problem implicitly. An incremental implementation of MMC can be found in [14].

3 Linear Discriminant Analysis

For convenience, we present in Table 1 the important notations used in the paper.

This section gives a brief review of classical LDA, as well as its three extensions: regularized LDA, PCA+LDA, and LDA/GSVD.

Given a data matrix $A \in \mathbb{R}^{d \times n}$, we consider finding a linear transformation $G \in \mathbb{R}^{d \times \ell}$ that maps each column a_j, for $1 \leq j \leq n$, of A in the d-dimensional space to a vector $y_j = G^T a_j$ in the ℓ-dimensional space.

Classical LDA aims to find the transformation G such that class structure of the original high-dimensional space is

TABLE 1
Notations

Notations	Descriptions	Notations	Descriptions
n	number of training data points	n_i	number of data points in i-th class
d	dimension of the training data	c	number of classes
G	transformation matrix	A	data matrix
A_i	data matrix of the i-th class	S_b	between-class scatter matrix
S_w	within-class scatter matrix	C	centroid matrix
m	global centroid of the training set	m_i	centroid of the i-th class

preserved in the reduced space. Let the data matrix A be partitioned into c classes as $A = [A_1, A_2, \cdots, A_k]$, where $A_i \in \mathbb{R}^{d \times n_i}$, and $\sum_{i=1}^{c} n_i = n$.

Let I_i be the set of column indices that belong to the ith class, i.e., a_j, for $j \in I_i$, belongs to the ith class.

In general, if each class is tightly grouped, but well-separated from the other classes, the quality of the cluster is considered to be high. In discriminant analysis, two scatter matrices, *within-class* and *between-class* scatter matrices, are defined to quantify the quality of the cluster, as follows [17]:

$$S_w = \sum_{i=1}^{c} \sum_{j \in I_i} (a_j - m_i)(a_j - m_i)^T,$$

$$S_b = \sum_{i=1}^{c} \sum_{j \in I_i} (m_i - m)(m_i - m)^T = \sum_{i=1}^{c} n_i (m_i - m)(m_i - m)^T,$$

where m_i is the *centroid* of the ith class and m is the *global centroid*.

Define the matrices

$$H_w = [A_1 - m_1 \cdot e_1^T, \cdots, A_c - m_c \cdot e_c^T] \in \mathbb{R}^{d \times n}, \qquad (1)$$

$$H_b = [\sqrt{n_1}(m_1 - m), \cdots, \sqrt{n_c}(m_c - m)] \in \mathbb{R}^{d \times c}, \qquad (2)$$

where $e_i = (1, \cdots, 1)^T \in \mathbb{R}^{n_i}$.

Then, the scatter matrices S_w and S_b can be expressed as $S_w = H_w H_w^T$, $S_b = H_b H_b^T$. The traces of the two scatter matrices can be computed as follows:

$$\text{trace}\,(S_w) = \sum_{i=1}^{c} \sum_{j \in I_i} ||a_j - m_i||^2,$$

$$\text{trace}\,(S_b) \; = \sum_{i=1}^{c} n_i ||m_i - m||^2.$$

Hence, $\text{trace}(S_w)$ measures the closeness of the vectors within classes, while $\text{trace}(S_b)$ measures the separation between classes.

In the lower-dimensional space resulting from the linear transformation G, the within-class and between-class scatter matrices become

$$S_w^L = (G^T H_w)(G^T H_w)^T = G^T S_w G,$$

$$S_b^L = (G^T H_b)(G^T H_b)^T = G^T S_b G.$$

An optimal transformation G would maximize $\text{trace}(S_b^L)$ and minimize $\text{trace}(S_w^L)$. A common optimization in classical LDA [17] is to compute

$$G = \arg \max_{g_i^T S_w g_j = 0, \forall i \neq j} \text{trace}\Big((G^T S_w G)^{-1} (G^T S_b G)\Big), \qquad (3)$$

where g_i is the ith column of G.

The solution to the optimization in (3) can be obtained by solving the eigenvalue problem on $S_w^{-1} S_b$, if S_w is nonsingular, or on $S_b^{-1} S_w$, if S_b is nonsingular. There are at most $c - 1$ eigenvectors corresponding to nonzero eigenvalues since the rank of the matrix S_b is bounded from above by $c - 1$. Therefore, the reduced dimension by classical LDA is at most $c - 1$. A stable way to solve this eigenvalue problem is to apply SVD on the scatter matrices. Details on this can be found in [34].

Classical LDA is equivalent to maximum likelihood classification assuming normal distribution for each class with the common covariance matrix [21]. When each class has more complex structure, classical LDA may fail. Generalization of LDA by fitting Gaussian mixtures to each class has been studied in [20].

Classical LDA requires that one of the scatter matrices be nonsingular. For many applications involving undersampled data, where the data dimension is much greater than the number of data items, such as in text and image retrieval, all scatter matrices are singular. Classical LDA is thus not applicable. This is the so-called *singularity* or *undersampled* problem. To cope with this probelm, several methods, including two-stage PCA+LDA, regularized LDA, and LDA/GSVD have been proposed in the past.

A common way to deal with the singularity problem is to apply an intermediate dimension reduction stage, such as PCA, to reduce the dimension of the original data before classical LDA is applied. The algorithm is known as *PCA+LDA*, or *subspace LDA*. In this two-stage PCA+LDA algorithm, the discriminant stage is preceded by a dimension reduction stage using PCA. A limitation of this approach is that the optimal value of the reduced dimension for PCA is difficult to determine.

Another common way to deal with the singularity problem is to add some constant value to the diagonal elements of S_w, as $S_w + \mu I_d$, for some $\mu > 0$, where I_d is an identity matrix [16]. It is easy to check that $S_w + \mu I_d$ is positive definite, hence, nonsingular. This approach is called *regularized LDA* (RLDA). A limitation of RLDA is that the optimal value of the parameter μ is difficult to determine. Cross-validation is commonly applied for estimating the optimal μ [25].

The LDA/GSVD algorithm in [22], [36] is a more recent approach. A new criterion for generalized LDA is presented in [36]. The inversion of the matrix S_w is avoided by applying the Generalized Singular Value Decomposition (GSVD). LDA/GSVD computes the solution exactly without losing any information. However, one limitation of this method is the high computational cost of GSVD, which limits its applicability for large data sets, such as image and text data.

4 Batch IDR/QR

In this section, we present the batch implementation of the IDR/QR algorithm. This algorithm has two stages: The first stage maximizes the separation between different classes via QR Decomposition [18]. The second stage addresses the issue of minimizing the within-class distance, while keeping low time/space complexity. Ignoring the issue of minimizing within-class distance, the first stage can be used independently as a dimension reduction algorithm.

The first stage of IDR/QR aims to solve the following optimization problem:

$$G = \arg \max_{G^T G = I} \operatorname{trace}(G^T S_b G). \tag{4}$$

Note that this optimization only addresses the issue of maximizing the between-class distance. The solution can be obtained by solving the eigenvalue problem on S_b.

Theorem 1. *Let $S_b = U \Sigma U^T$ be the SVD of S_b, where $U \in \mathbb{R}^{d \times q}$ has orthonormal columns, $\Sigma = \operatorname{diag}(\sigma_1, \cdots, \sigma_q) \in \mathbb{R}^{q \times q}$ is diagonal, and $q = \operatorname{rank}(S_b)$. Then, $G^* = U$ solves the optimization problem in (4).*

Proof. By the property of the trace, we have

$$\begin{aligned}\operatorname{trace}(G^T S_b G) &\leq \operatorname{trace}(S_b) = \operatorname{trace}(U \Sigma U^T) \\ &= \operatorname{trace}(\Sigma U^T U) = \sum_{i=1}^{q} \sigma_i,\end{aligned}$$

where the first inequality follows from Lemma 1 in the Appendix. Thus, the optimization in (4) is bounded from above by $\sum_{i=1}^{q} \sigma_i$.

On the other hand,

$$\operatorname{trace}((G^*)^T S_b G^*) = \operatorname{trace}(U^T U \Sigma U^T U) = \operatorname{trace}(\Sigma) = \sum_{i=1}^{q} \sigma_i,$$

that is, the upper bound is achieved with $G^* = U$. This completes the proof of the theorem. □

The solution can also be obtained through QR Decomposition on the centroid matrix C, which is the so-called Orthogonal Centroid Method (OCM) [30], where

$$C = [m_1, m_2, \cdots, m_c] \tag{5}$$

consists of the c centroids. The result is summarized as follows:

Theorem 2. *Let $C = QR$ be the QR Decomposition of C, where $Q \in \mathbb{R}^{d \times c}$ has orthonormal columns and $R \in \mathbb{R}^{c \times c}$ is upper triangular. Then,*

$$G^* = QM, \tag{6}$$

for any orthogonal matrix M, solves the optimization problem in (4).

Proof. It is easy to check that $H_b = CE$, where $E \in \mathbb{R}^{c \times c}$ and the ith column of E is

$$\sqrt{n_i}(0, \cdots, 0, 1, 0, \cdots, 0)^T + \frac{\sqrt{n_i}}{n}(n_1, n_2, \cdots, n_c)^T.$$

Let $C = QR$ be the QR Decomposition of C. Then,

$$S_b = H_b H_b^T = C E E^T C^T = Q(R E E^T R^T) Q^T = Q \hat{E} Q^T,$$

where $\hat{E} = R E E^T R^T$.

For any G with orthonormal columns, it is clear that

$$\begin{aligned}\operatorname{trace}\big(G^T S_b G\big) &\leq \operatorname{trace}(S_b) = \operatorname{trace}\big(Q \hat{E} Q^T\big) \\ &= \operatorname{trace}\big(\hat{E} Q^T Q\big) = \operatorname{trace}\big(\hat{E}\big),\end{aligned}$$

where the the first inequality follows from Lemma 1 in the Appendix. Thus, $\operatorname{trace}\big(\hat{E}\big)$ is an upper bound for the optimization in (4). Next, we show that the upper bound is achieved by choosing $G^* = QM$ for any orthogonal M, as in (6).

By the property of the trace and the fact that Q has orthonormal columns, we have

$$\begin{aligned}\operatorname{trace}\Big((G^*)^T S_b G^*\Big) &= \operatorname{trace}\big(M^T Q^T Q \hat{E} Q^T Q M\big) \\ &= \operatorname{trace}\big(M^T \hat{E} M\big) = \operatorname{trace}\big(\hat{E} M M^T\big) \\ &= \operatorname{trace}\big(\hat{E}\big).\end{aligned}$$

This completes the proof of the theorem. □

Note the choice of orthogonal matrix M is arbitrary since $\operatorname{trace}(G^T S_b G) = \operatorname{trace}(M^T G^T S_b G M)$, for any orthogonal matrix M. In the OCM method [30], M is set to be the identity matrix for simplicity.

Remark 1. Note that from Theorems 1 and 2, both the matrix U based on the eigen-decomposition of S_b and the matrix Q based on the QR Decomposition of C solve the optimization problem in (4). In most applications, the c centroids in the data set are linearly independent. In this case, the column dimension of the matrix U in Theorem 1 is $q = \operatorname{rank}(S_b) = c - 1$, which is one less than the column dimension of the matrix Q in Theorem 2. Experiments show that both solutions are comparable in terms of classification accuracy. However, the solution based on QR Decomposition of C is preferred, when incremental updating is required. This is because of the key observation that when a new data item is inserted, at most one column of the centroid matrix C is modified, which leads to the efficient updating of the QR Decomposition of the centroid matrix. Details can be found in Section 5.

The second stage of IDR/QR refines the first stage by addressing the issue of minimizing the within-class distance. It incorporates the within-class scatter information by applying a relaxation scheme on M in (6) (relaxing M from an orthogonal matrix to an arbitrary matrix). Note that the trace value in (3) is the same for an arbitrary nonsingular M; however, the constraints in (3) will not be

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Algorithm 1: Batch IDR/QR
Input:    data matrix $A$;
Output: optimal transformation matrix $G$;
/* Stage I: */
1. Construct centroid matrix $C$;
2. Compute QR Decomposition of $C$ as $C = QR$, where $Q \in \mathbb{R}^{d \times c}$, $R \in \mathbb{R}^{c \times c}$;
/* Stage II: */
3. $Z \leftarrow H_w^T Q$;
4. $Y \leftarrow H_b^T Q$;
5. $W \leftarrow Z^T Z$; /*Reduced within-class scatter matrix*/
6. $B \leftarrow Y^T Y$; /*Reduced between-class scatter matrix*/
7. Compute the $c$ eigenvectors $\phi_i$ of $(W + \mu I_c)^{-1} B$ with decreasing eigenvalues;
8. $G \leftarrow QM$, where $M = [\phi_1, \cdots, \phi_c]$.
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Fig. 1. Algorithm 1: Batch IDR/QR.

satisfied for arbitrary M. In the second stage of IDR/QR, we look for a transformation matrix G such that $G = QM$, for some M. Note that M is not required to be orthogonal. The original problem on computing G is equivalent to computing M. Since

$$G^T S_b G = M^T (Q^T S_b Q) M,$$
$$G^T S_w G = M^T (Q^T S_w Q) M,$$

the original optimization on finding optimal G is equivalent to finding M, with $B = Q^T S_b Q$ and $W = Q^T S_w Q$ as the reduced between-class and within-class scatter matrices, respectively. Note that B has much smaller size than the original scatter matrix S_b (similarly for W).

The optimal M can be computed efficiently using many existing LDA-based methods since we are dealing with matrices B and W of much smaller size, i.e., $c \times c$. A key observation is that the singularity problem of W will not be as severe as the original S_w since W has a much smaller size than S_w. We can compute optimal M by simply applying regularized LDA; that is, we compute M by solving a small eigenvalue problem on $(W + \mu I_c)^{-1} B$, for some positive constant μ. Extensive experiments show that the solution is insensitive to the choice of μ, due to the small size of W. The pseudocode for this algorithm is given in **Algorithm 1** as shown in Fig. 1.

4.1 Time and Space Complexity

We close this section by analyzing the time and space complexity of the batch IDR/QR algorithm.

It takes $O(dn)$ for the formation of the centroid matrix C in Line 1 of Algorithm 1 (Fig. 1). The complexity of doing QR Decomposition in Line 2 is $O(c^2 d)$ [18]. Lines 3 and 4 take $O(ndc)$ and $O(dc^2)$, respectively, for matrix multiplications. It then takes $O(c^2 n)$ and $O(c^3)$ for matrix multiplications in Lines 5 and 6, respectively. Line 7 computes the eigen-decomposition of a $c \times c$ matrix, hence, takes $O(c^3)$ [18]. The matrix multiplication in Line 8 takes $O(dc^2)$.

Note that the dimension, d, and the number, n, of points are usually much larger than the number, c, of classes. Thus, the most expensive step in **Algorithm 1** (Fig. 1) is Line 3, which takes $O(ndc)$ time. Therefore, the time complexity of IDR/QR is linear in the number of points, linear in the number of classes, and linear in the dimension of the data set.

It is clear that only the c centroids are required to reside in the main memory; hence, the space complexity of IDR/QR is $O(dc)$. Table 2 lists the time and space complexity of several dimension reduction algorithms discussed in this paper. It is clear from the table that IDR/QR is more efficient than other LDA-based methods (except OCM).

TABLE 2
Complexity Comparison: n Is the Number of Training Data Points, d Is the Dimension, and c Is the Number of Classes

Methods	Time Complexity	Space Complexity
IDR/QR	$O(ndc)$	$O(dc)$
PCA+LDA	$O(n^2 d)$	$O(nd)$
LDA/GSVD	$O((n+c)^2 d)$	$O(nd)$
OCM	$O(nd + c^2 d)$	$O(dc)$
PCA	$O(n^2 d)$	$O(nd)$

5 Incremental IDR/QR

The incremental implementation of the IDR/QR algorithm is discussed in detail in this section. We will adopt the following convention: For any variable X, its updated version after the insertion of a new instance is denoted by $\tilde{X}$. For example, the number, n_i, of elements in the ith class will be changed to $\tilde{n}_i$, while centroid m_i will be changed to $\tilde{m}_i$.

With the insertion of a new instance, the centroid matrix C, H_w, and H_b will change accordingly, as well as W and B. The incremental updating in IDR/QR proceeds in three steps:

1. QR-updating of centroid matrix $C = [m_1, \cdots, m_k]$ in Line 2 of **Algorithm 1** (Fig. 1),
2. updating of reduced within-class scatter matrix W in Line 5, and
3. updating of reduced between-class scatter matrix B in Line 6.

Let x be a new instance inserted; let x belong to the ith class. Without loss of generality, let us assume that we have data from the first to the kth class, just before x is inserted. In general, this can be done by switching the class labels between different classes. In the rest of this section, we consider the incremental updating in IDR/QR in two distinct cases: 1) x belongs to an existing class, i.e., $i \leq k$; 2) x belongs to a new class, i.e., $i > k$. As will be seen later, the techniques for these two cases are quite different.

5.1 Insertion of a New Instance from an Existing Class ($i \leq k$)

Recall that we have data from the first to kth classes, when a new instance x is being inserted. Since x belongs to the ith class, with $1 \leq i \leq k$, the insertion of x will not create a new class. In this section, we show how to do the incremental updating in three steps.

5.1.1 Step 1: QR-Updating of Centroid Matrix C

Since the new instance x belongs to the ith class,

$$\tilde{C} = [m_1, \cdots, m_i + f, \cdots, m_k],$$

where $f = \frac{x - m_i}{\tilde{n}_i}$, and $\tilde{n}_i = n_i + 1$. Hence, $\tilde{C}$ can be rewritten as $\tilde{C} = C + f \cdot g^T$, for $g = (0, \cdots, 1, \cdots, 0)^T$, where the 1 appears at the ith position.

The problem of QR-updating of the centroid matrix C can be formulated as follows: Given the QR Decomposition of the centroid matrix $C = QR$, for $Q \in \mathbb{R}^{d \times k}$ and $R \in \mathbb{R}^{k \times k}$, compute the QR Decomposition of $\tilde{C}$.

Since $\tilde{C} = C + f \cdot g^T$, the QR-updating of the centroid matrix, C, can be formulated as a rank-one QR-updating. However, the algorithm in [18] cannot be directly applied since it requires the complete QR Decomposition, i.e., the matrix Q is square, while in our case, we use the skinny QR Decomposition, i.e., Q is rectangular. Instead, we apply a small variation of the algorithm in [9] via the following two-stage QR-updating: 1) A complete rank-one updating as in [18] on a small matrix and 2) a QR-updating by an insertion of a new row. Details are given below.

Partition f into two parts: the projection onto the orthogonal basis Q and its orthogonal complement. Mathematically, f can be partitioned into $f = QQ^T f + (I - QQ^T) f$. It is easy to check that $Q^T(I - QQ^T)f = 0$, i.e., $(I - QQ^T)f$ is orthogonal to, or lies in the orthogonal complement of, the subspace spanned by the columns of Q. It follows that

$$\begin{aligned}\tilde{C} &= C + f \cdot g^T \\ &= QR + QQ^T f \cdot g^T + (I - QQ^T) f \cdot g^T \\ &= Q(R + f_1 \cdot g^T) + f_2 \cdot g^T,\end{aligned}$$

where $f_1 = Q^T f$, $f_2 = (I - QQ^T) f$. Next, we show how to compute the QR Decomposition of $\tilde{C}$ in two stages: The first stage updates the QR Decomposition of $Q(R + f_1 \cdot g^T)$. It corresponds to a rank-one updating and can be done at $O(kd)$ [18]. This results in the updated QR Decomposition as $Q(R + f_1 \cdot g^T) = Q_1 R_1$, where $Q_1 = QP_1$, and $P_1 \in \mathbb{R}^{k \times k}$ is orthogonal.

Assume $\| f_2 \| \neq 0$. Denote $q = \frac{f_2}{\|f_2\|}$. Since q is orthogonal to the subspace spanned by the columns of Q, it is also orthogonal to the subspace spanned by the columns of $Q_1 = QP_1$, i.e., $[Q_1, q]$ has orthonormal columns.

The second stage computes QR-updating of

$$\tilde{C} = [Q_1, q] \begin{pmatrix} R_1 \\ ||f_2|| g^T \end{pmatrix},$$

which corresponds to the case that $\| f_2 \| g^T$ is inserted as a new row. This stage can be done at $O(dk)$ [18]. The updated QR Decomposition is

$$[Q_1, q] \begin{pmatrix} R_1 \\ \| f_2 \| g^T \end{pmatrix} = [\tilde{Q}, \tilde{q}] \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} = \tilde{Q}\tilde{R},$$

where $[\tilde{Q}, \tilde{q}] = [Q_1, q] P_2$, for some orthogonal matrix P_2.

Combining both stages, we have

$$\tilde{C} = Q_1 R_1 + \| f_2 \| q \cdot g^T = [Q_1, q] \begin{pmatrix} R_1 \\ \| f_2 \| g^T \end{pmatrix} = \tilde{Q}\tilde{R}$$

as the updated QR Decomposition of $\tilde{C}$, assuming $\| f_2 \| \neq 0$. If $\| f_2 \| = 0$, then $\tilde{C} = Q_1 R_1$ is the updated QR Decomposition of $\tilde{C}$. Note that f_2 can be computed efficiently as $f_2 = f - (Q(Q^T f))$ by doing matrix-vector multiplication twice. Hence, the total time complexity for the QR-updating of the centroid matrix C is $O(dk)$.

5.1.2 Step 2: Updating of W

Next, we consider the updating of the reduced within-class scatter matrix $W = Q^T H_w H_w^T Q$ (Line 5 of **Algorithm 1** in Fig. 1). Let $\tilde{W} = \tilde{Q}^T \tilde{H}_w \tilde{H}_w^T \tilde{Q}$ be its updated version.

Note that $H_w = [A_1 - m_1 \cdot e_1^T, \cdots, A_k - m_k \cdot e_k^T] \in \mathbb{R}^{d \times n}$. Its updated version $\tilde{H}_w$ differs from H_w in the ith block. Let the ith block of H_w be $H_i = A_i - m_i \cdot e_i^T$. Then, the ith block of its updated version $\tilde{H}_w$ is

$$\begin{aligned}\tilde{H}_i &= \tilde{A}_i - \tilde{m}_i \cdot \tilde{e}_i^T = [A_i, x] - \tilde{m}_i \cdot \tilde{e}_i^T \\ &= [A_i - m_i \cdot e_i^T, x - m_i] - (\tilde{m}_i - m_i) \cdot \tilde{e}_i^T \\ &= [H_i, u] - v \cdot \tilde{e}_i^T,\end{aligned} \tag{7}$$

where $u = x - m_i$, $v = \tilde{m}_i - m_i$, and $\tilde{e}_i = \begin{pmatrix} e_i \\ 1 \end{pmatrix} \in \mathbb{R}^{n_i+1}$.

The product $\tilde{H}_i\tilde{H}_i^T$ can be computed as

$$\begin{aligned}\tilde{H}_i\tilde{H}_i^T &= ([H_i,u]-v\cdot\tilde{e}_i^T)([H_i,u]-v\cdot\tilde{e}_i^T)^T\\ &= [H_i,u]\begin{pmatrix}H_i^T\\ u^T\end{pmatrix} - v\cdot\tilde{e}_i^T\begin{pmatrix}H_i^T\\ u^T\end{pmatrix}\\ &\quad-[H_i,u]\tilde{e}_i\cdot v^T + (v\cdot\tilde{e}_i^T)(\tilde{e}_i\cdot v^T)\\ &= H_iH_i^T + u\cdot u^T - v\cdot u^T - u\cdot v^T + (n_i+1)v\cdot v^T\\ &= H_iH_i^T + (u-v)\cdot(u-v)^T + n_iv\cdot v^T,\end{aligned} \tag{8}$$

where the third equality follows, since $(H_i,u)\tilde{e}_i = \sum_{j\in I_i}(a_j - m_i) + u = u$ and $(v\cdot\tilde{e}_i^T)(\tilde{e}_i\cdot v^T) = vv^T(\tilde{e}_i^T\cdot\tilde{e}_i) = (n_i+1)vv^T$.

Since $H_wH_w^T = \sum_{j=1}^k H_jH_j^T$, we have

$$\begin{aligned}\tilde{H}_w\tilde{H}_w^T &= \sum_{j=1}^k \tilde{H}_j\tilde{H}_j^T\\ &= \sum_{1\le j\le k, j\ne i}\tilde{H}_j\tilde{H}_j^T + \tilde{H}_i\tilde{H}_i^T\\ &= \sum_{j=1}^k H_jH_j^T + (u-v)\cdot(u-v)^T + n_iv\cdot v^T.\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{W} &= \tilde{Q}^T\tilde{H}_w\tilde{H}_w^T\tilde{Q}\\ &= \tilde{Q}^TH_wH_w^T\tilde{Q} + \tilde{Q}^T(u-v)\cdot(u-v)^T\tilde{Q} + n_i\tilde{Q}^Tv\cdot v^T\tilde{Q}\\ &= \tilde{Q}^TH_wH_w^T\tilde{Q} + (\tilde{u}-\tilde{v})\cdot(\tilde{u}-\tilde{v})^T + n_i\tilde{v}\cdot\tilde{v}^T\\ &\approx QH_wH_w^TQ + (\tilde{u}-\tilde{v})\cdot(\tilde{u}-\tilde{v})^T + n_i\tilde{v}\cdot\tilde{v}^T\\ &= W + (\tilde{u}-\tilde{v})\cdot(\tilde{u}-\tilde{v})^T + n_i\tilde{v}\cdot\tilde{v}^T,\end{aligned} \tag{9}$$

where $\tilde{u} = \tilde{Q}^Tu$ and $\tilde{v} = \tilde{Q}^Tv$. The assumption of the approximation in (9) is that the updated $\tilde{Q}$ with the insertion of a new instance is close to Q.

The computation of $\tilde{u}$ and $\tilde{v}$ takes $O(dk)$ time. Thus, the computation for updating W takes $O(dk)$.

5.1.3 Step 3: Updating of B

Finally, let us consider the updating of the reduced between-class scatter matrix $B = Q^TH_bH_b^TQ$ (Line 6 of **Algorithm 1** in Fig. 1). Its updated version is $B = \tilde{Q}^T\tilde{H}_b\tilde{H}_b^T\tilde{Q}$.

The key observation for efficient updating of B is that

$$\tilde{H}_b = [\sqrt{\tilde{n}_1}(\tilde{m}_1 - \tilde{m}),\cdots,\sqrt{\tilde{n}_k}(\tilde{m}_k-\tilde{m})]$$

can be rewritten as

$$\tilde{H}_b = [\tilde{m}_1,\tilde{m}_2,\cdots,\tilde{m}_k,\tilde{m}]F = [\tilde{C},\tilde{m}]F,$$

where

$$F = \begin{pmatrix}D\\ -h^T\end{pmatrix},$$

$D = \mathrm{diag}(\sqrt{\tilde{\mathrm{n}}_1},\cdots,\sqrt{\tilde{\mathrm{n}}_\mathrm{k}})$, and $h = [\sqrt{\tilde{n}_1},\cdots,\sqrt{\tilde{n}_k}]^T$.

By the updated QR Decomposition $\tilde{C} = \tilde{Q}\tilde{R}$, we have

$$\tilde{Q}^T\tilde{H}_b = [\tilde{Q}^T\tilde{C},\tilde{Q}^T\tilde{m}]F = [\tilde{R},\tilde{Q}^T\tilde{m}]F = \tilde{R}D - \tilde{Q}^T\tilde{m}\cdot h^T.$$

It is easy to check that $\tilde{m} = \frac{1}{\tilde{n}}\tilde{C}\cdot r$, where $r = (\tilde{n}_1,\cdots,\tilde{n}_k)^T$. Hence, $\tilde{Q}^T\tilde{m} = \tilde{Q}^T\frac{1}{\tilde{n}}\tilde{C}\cdot r = \frac{1}{\tilde{n}}\tilde{R}\cdot r$. It follows that

$$\begin{aligned}\tilde{B} &= \tilde{Q}^T\tilde{H}_b\tilde{H}_b^T\tilde{Q} = (\tilde{R}D - \tilde{Q}^T\tilde{m}\cdot h^T)\cdot(\tilde{R}D - \tilde{Q}^T\tilde{m}\cdot h^T)^T\\ &= \left(\tilde{R}D - \left(\frac{1}{\tilde{n}}\tilde{R}\cdot r\right)\cdot h^T\right)\left(\tilde{R}D - \left(\frac{1}{\tilde{n}}\tilde{R}\cdot r\right)\cdot h^T\right)^T.\end{aligned}$$

Therefore, it takes $O(k^3)$ time for updating B.

Overall, the total time for QR-updating of C and updating of W and B with the insertion of a new instance from an existing class is $O(dk + k^3)$. The pseudocode is given in **Algorithm 2** in Fig. 2.

5.2 Insertion of a New Instance from a New Class ($i > k$)

Recall that we have data from the first to kth classes, upon the insertion of x. Since x belongs to ith class, with $i > k$, the insertion of x will result in a new class. Without loss of generality, let us assume $i = k+1$. Hence, the $(k+1)\mathrm{th}$ centroid $\tilde{m}_{k+1} = x$. Then, the updated centroid matrix $\tilde{C} = [m_1, m_2,\cdots,m_k,x] = [C,x]$. In the following, we focus on the case when x does not lie in the space spanned by the k centroids $\{m_i\}_{i=1}^k$.

5.2.1 Step 1: QR-Updating of Centroid Matrix C

Given the QR Decomposition $C = QR$, it is straightforward to compute the QR Decomposition of $\tilde{C}$ as $\tilde{C} = \tilde{Q}\tilde{R}$ by the Gram-Schmidt procedure [18], where $\tilde{Q} = [Q,q]$, for some q. The time complexity for this step is $O(dk)$.

5.2.2 Step 2: Updating of W

With the insertion of x from a new class $(k+1)$, the $(k+1)\mathrm{th}$ block $\tilde{H}_{k+1}$ is created, while H_j, for $j = 1,\cdots,k$ keep unchanged. It is easy to check that $\tilde{H}_{k+1} = 0$. It follows that $\tilde{H}_w\tilde{H}_w^T = H_wH_w^T$. Hence,

$$\begin{aligned}\tilde{W} &= \tilde{Q}^T\tilde{H}_w\tilde{H}_w^T\tilde{Q} = \tilde{Q}^TH_wH_w^T\tilde{Q} = [Q,q]^TH_wH_w^T[Q,q]\\ &\approx \begin{pmatrix}Q^TH_wH_w^TQ & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}W & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

The assumption in the above approximation is that W is the dominant part in $\tilde{W}$.

5.2.3 Step 3: Updating of B

The updating of B follows the same idea as in the previous case. Note that

$$\tilde{H}_b = \left[\sqrt{\tilde{n}_1}(\tilde{m}_1-\tilde{m}),\cdots,\sqrt{\tilde{n}_{k+1}}(\tilde{m}_{k+1}-\tilde{m})\right]$$

can be rewritten as

$$\tilde{H}_b = [\tilde{m}_1,\tilde{m}_2,\cdots,\tilde{m}_{k+1},\tilde{m}]F,$$

where the matrix

$$F = \begin{pmatrix}D\\ -h^T\end{pmatrix}$$

and D is an diagonal matrix $D = \mathrm{diag}(\sqrt{\tilde{\mathrm{n}}_1},\cdots,\sqrt{\tilde{\mathrm{n}}_{\mathrm{k+1}}})$ and $h = [\sqrt{\tilde{n}_1},\cdots,\sqrt{\tilde{n}_{k+1}}]^T$.

By the updated QR Decomposition $\tilde{C} = \tilde{Q}\tilde{R}$, we have

$$\begin{aligned}\tilde{Q}^T\tilde{H}_b &= \tilde{Q}^T[\tilde{C},\tilde{m}]F = [\tilde{Q}^T\tilde{C},\tilde{Q}^T\tilde{m}]F\\ &= [\tilde{R},\tilde{Q}^T\tilde{m}]F = \tilde{R}D - \tilde{Q}^T\tilde{m}\cdot h^T.\end{aligned}$$

```
Algorithm 2: Updating Existing Class
Input: centroid matrix $C = [m_1, m_2, \cdots, m_k]$, its QR Decomposition $C = QR$, the matrix $W$, the size $n_j$ of
       the $j$-th class for each $j$, and a new point $x$ from the $i$-th class, $i \leq k$
Output: updated matrix $\tilde{W}$, updated centroid matrix $\tilde{C}$, its QR Decomposition $\tilde{C} = \tilde{Q}\tilde{R}$, and updated matrix $\tilde{B}$;
1. $\tilde{n}_j \leftarrow n_j$, for $j \neq i$; $\tilde{n}_i \leftarrow n_i + 1$; $f \leftarrow \frac{x - m_i}{\tilde{n}_i}$;
2. $\tilde{m}_i \leftarrow m_i + f$; $\tilde{m}_j \leftarrow m_j$, for each $j \neq i$;
3. $\tilde{C} \leftarrow [\tilde{m}_1, \cdots, \tilde{m}_i, \cdots, \tilde{m}_k]$;
4. $f_1 \leftarrow Q^T f$; $f_2 \leftarrow (I - QQ^T)f$;
5. do rank-one QR-updating of $Q(R + f_1 \cdot g^T)$ as $Q(R + f_1 \cdot g^T) = Q_1 R_1$;
6. if $||f_2|| = 0$
7.     $\tilde{Q} \leftarrow Q_1$; $\tilde{R} \leftarrow R_1$;
8. else
9.     $q \leftarrow \frac{(I-QQ^T)f}{||(I-QQ^T)f||}$; $g \leftarrow (0, \cdots, 1, \cdots, 0)^T$;
10.    do QR-updating of $[Q_1, q]\begin{pmatrix} R_1 \\ ||f_2||g^T \end{pmatrix}$ as $[Q_1, q]\begin{pmatrix} R_1 \\ ||f_2||g^T \end{pmatrix} = Q_2 R_2$;
11.    $\tilde{Q} \leftarrow Q_2$; $\tilde{R} \leftarrow R_2$;
12. endif
13. $u \leftarrow x - m_i$; $v \leftarrow \tilde{m}_i - m_i$;
14. $\tilde{u} \leftarrow \tilde{Q}^T u$; $\tilde{v} \leftarrow \tilde{Q}^T v$;
15. $\tilde{W} \leftarrow W + (\tilde{u} - \tilde{v})(\tilde{u} - \tilde{v})^T + n_i \tilde{v}\tilde{v}^T$;
16. $D \leftarrow \text{diag}(\sqrt{\tilde{n}_1}, \cdots, \sqrt{\tilde{n}_k})$; $h = [\sqrt{\tilde{n}_1}, \cdots, \sqrt{\tilde{n}_k}]^T$;
17. $r \leftarrow (\tilde{n}_1, \cdots, \tilde{n}_k)^T$; $\tilde{r} \leftarrow \frac{1}{\tilde{n}}\tilde{R} \cdot r$;
18. $\tilde{B} \leftarrow (\tilde{R}D - \tilde{r} \cdot h^T)(\tilde{R}D - \tilde{r} \cdot h^T)^T$;
```

Fig. 2. Algorithm 2: Updating existing class.

It is easy to check that $\tilde{m} = \frac{1}{\tilde{n}}\tilde{C} \cdot r$ where

$$r = (\tilde{n}_1, \cdots, \tilde{n}_{k+1})^T.$$

Hence, $\tilde{Q}^T\tilde{m} = \tilde{Q}^T \frac{1}{\tilde{n}}\tilde{C} \cdot r = \frac{1}{\tilde{n}}\tilde{R} \cdot r$.

Then, $\tilde{B}$ can be computed by similar arguments as in the previous case. Therefore, it takes $O(k^3)$ time for updating B.

Thus, the time for QR-updating of C and updating of W and B with the insertion of a new instance from a new class is $O(dk + k^3)$. The pseudocode is given in **Algorithm 3** as shown in Fig. 3.

5.3 Main Algorithm

With the above two incremental updating schemes, the incremental IDR/QR works as follows: For a given new instance x, determine whether it is from an existing class or belongs to a new class. If it is from an existing class, update the QR Decomposition of the centroid matrix C and W and B by applying **Algorithm 2** in Fig. 2; otherwise, update the QR Decomposition of the centroid matrix C and W and B by applying **Algorithm 3** in Fig. 3. The above procedure is repeated until all points are considered. With the final updated $\tilde{W}$ and $\tilde{B}$, we can compute the c eigenvectors $\{\phi_i\}_{i=1}^c$ of $(\tilde{W} + \mu I_c)^{-1}\tilde{B}$, and assign $[\phi_1, \cdots, \phi_c]$ to M. Then, the transformation $G = \tilde{Q}M$, assuming $\tilde{C} = \tilde{Q}\tilde{R}$ is the updated QR Decomposition.

The incremental IDR/QR proposed obeys the following general criteria for an *incremental learning* algorithm [31]:

1. It is able to learn new information from new data.

```
Algorithm 3: Updating New Class
Input: centroid matrix $C = [m_1, m_2, \cdots, m_k]$, its QR Decomposition $C = QR$, the size $n_j$ of the $j$-th class
       for each $j$, and a new point $x$ from the $(k+1)$-th class
Output: updated matrix $\tilde{W}$, updated centroid matrix $\tilde{C}$, its QR Decomposition $\tilde{C} = \tilde{Q}\tilde{R}$, and updated matrix $\tilde{B}$;
1. $\tilde{n}_j \leftarrow n_j$, for $j = 1, \cdots, k$; $\tilde{n}_{k+1} \leftarrow 1$; $\tilde{n} \leftarrow n + 1$;
2. do QR-updating of $\tilde{C} = [C, x]$ as $\tilde{C} = \tilde{Q}\tilde{R}$;
3. $\tilde{W} \leftarrow \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$;
4. $D \leftarrow \mathrm{diag}\left(\sqrt{\tilde{n}_1}, \cdots, \sqrt{\tilde{n}_{k+1}}\right)$; $h \leftarrow \left(\sqrt{\tilde{n}_1}, \cdots, \sqrt{\tilde{n}_{k+1}}\right)^T$;
5. $r \leftarrow (\tilde{n}_1, \cdots, \tilde{n}_{k+1})^T$; $\tilde{r} \leftarrow \frac{1}{\tilde{n}}\tilde{R}r$;
6. $\tilde{B} \leftarrow (\tilde{R}D - \tilde{r} \cdot h^T)(\tilde{R}D - \tilde{r} \cdot h^T)^T$;
```

Fig. 3. Algorithm 3: Updating new class.

2. It does not require access to the original data.
3. It preserves previously acquired knowledge.
4. It is able to accommodate new classes that may be introduced with new data.

6 Empirical Evaluation

In this section, we evaluate both the batch version and the incremental version of the IDR/QR algorithm. The performance is mainly measured in terms of the classification error rate and execution time. In the experiment, we applied the K-Nearest Neighbor (K-NN) method [12] as the classification algorithm and classification error rates are estimated by 10-fold cross validation.

Experimental Platform. All experiments were performed on a PC with a P4 1.8GHz CPU and 1GB main memory running the Linux operating system.

Experimental Data Sets. Our experiments were performed on the following four real-world data sets, which are from two different application domains, including face recognition and text retrieval. Some characteristics of these data sets are shown in Table 3:

1. AR[1] is a popular face image data set [29]. The face images in AR contain a large area of occlusion, due to the presence of sun glasses and scarves, which leads to a relatively large within-class variance in the data set. In our experiments, we use a subset of the AR data set. This subset contains 1,638 face images of entire face identities (126). The image size of this subset is 768×576. We first crop the image from row 100 to 500, column 200 to 550, and then subsample the cropped images down to a size of $101 \times 88 = 8,888$.
2. ORL[2] is another popular face image data set, which includes 40 face individuals, i.e., 40 classes. The face images in ORL only contain pose variation, and are perfectly centralized/localized. The image size of ORL is $92 \times 112 = 10,304$. All dimensions (10,304 in number) are used to test our dimension reduction algorithms.
3. tr41 document data set is derived from the TREC-5, TREC-6, and TREC-7 collections.[3]
4. re1 document data set is derived from *Reuters-21578* text categorization test collection Distribution 1.0.[4]

Both document data sets are from [37], where a stop-list is used to remove common words, and the words are stemmed using Porter's suffix-stripping algorithm [32]. Moreover, any term that occurs in fewer than two documents was eliminated. Finally, the *tf-idf* weighting scheme [33] is used for encoding the document collection with a term-document matrix.

6.1 The Effect of Regularization Parameter μ on Batch IDR/QR

In this experiment, we study the effect of the regularization parameter μ on IDR/QR. Note that μ is used in the second stage of IDR/QR as the regularization term (See Line 7 of **Algorithm 1** in Fig. 1). The result is summarized in Fig. 4, where the horizontal axis denotes the value of the regularization parameter μ and the vertical axis denotes the classification error rate of batch IDR/QR. 1-NN is used to compute the classification error rate. It is clear from Fig. 4 that the performance of batch IDR/QR is insensitive to the choice of μ. This is likely due to the fact that the reduced within-class and between-class scatter matrices in Lines 5 and 6 of **Algorithm 1** in Fig. 1 are of small size ($k \times k$). In the following experiment, we simply set $\mu = 0.5$.

6.2 The Performance of Batch IDR/QR

In this experiment, we compare the performance of the batch IDR/QR with several other dimension reduction algorithms

1. http://rvl1.ecn.purdue.edu/~aleix/aleix_face_DB.html.
2. http://www.uk.research.att.com/facedatabase.html.
3. http://trec.nist.gov.
4. http://www.research.att.com/~lewis.

TABLE 3
Statistics for Our Test Data Sets

Data set	# of data points (n)	# of dimensions (d)	# of classes (c)
AR	1638	8888	126
ORL	400	10304	40
tr41	878	7454	10
re0	1504	2886	13

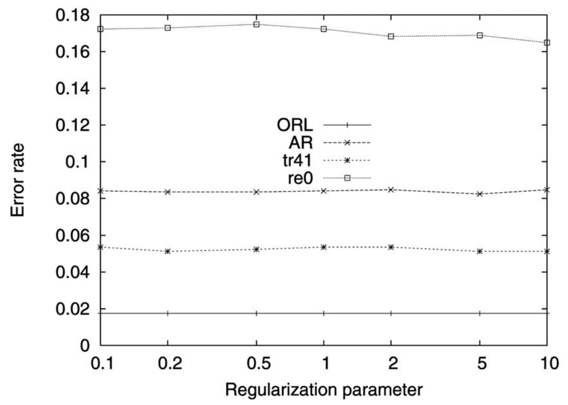


Fig. 4. The effect of regularization parameter μ on batch IDR/QR.

including PCA+LDA, LDA/GSVD, OCM, and PCA. Note that IDR/QR applies regularization to the reduced within-class scatter, i.e., $W + \mu I_c$. We chose $\mu = 0.5$ in our experiments, since it produced good overall results.

6.2.1 Classification Performance

Figs. 5 and 6 show the classification error rates on image and text document data sets, respectively, using five different dimension reduction algorithms. The main observations are as follows:

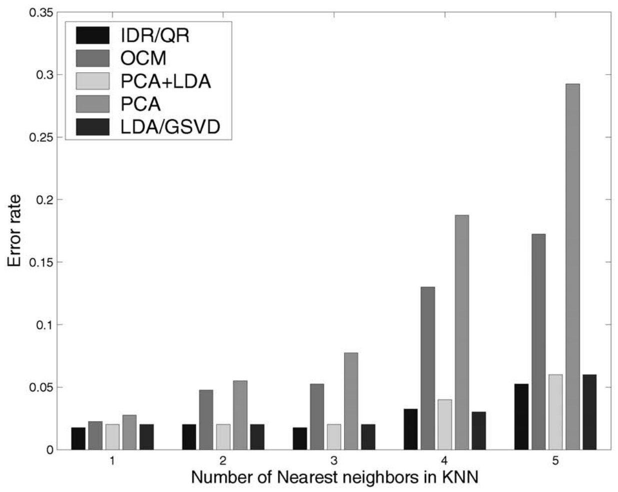
- The most interesting result is from the AR data set. We can observe that batch IDR/QR, PCA+LDA, and LDA/GSVD significantly outperform the other two dimension reduction algorithms, PCA and OCM, in terms of the classification error rate. Recall that the face images in the AR data set contain a large area of occlusion, which results in the large within-class variance in each class. The effort of minimizing of the within-class variance achieves distinct success in this situation. However, neither PCA nor OCM has the effort in minimizing the within-class variance. This explains why they have a poor classification performance on AR.
- Another interesting observation is that OCM performs well on text data sets. This observation is likely due to the fact that text data sets tend to have relatively small within-class variances. This observation suggests that OCM is a good choice, in practice, if the data is known to have small within-class variances.

6.2.2 Efficiency in Computing the Transformation

Fig. 7 shows the execution time (on a log-scale) of different tested methods for computing the transformation. Even with the log-scale presentation, we can still observe that the execution time for computing the transformation by IDR/QR or OCM is significantly smaller than that by PCA+LDA, LDA/GSVD, and PCA.

6.2.3 The Effect of Small Reduced Dimension

Here, we evaluate the effect of small reduced dimension on the classification error rate using the AR data set. Recall that the reduced dimension by the IDR/QR algorithm is c, where c is the number of classes in the data set. If the value c is large (such as AR, which contains 126 classes), the reduced representation may not be suitable for efficient indexing and retrieval. Since the reduced dimensions from IDR/QR are ordered by their discriminant power (see Line 7 of **Algorithm 1** in Fig. 1), an intuitive solution is to choose the first few dimensions in the reduced subspace from IDR/QR. The experimental results are shown in Fig. 8. As can be seen, the accuracy achieved by keeping the first 20 dimensions only is still sufficiently high.



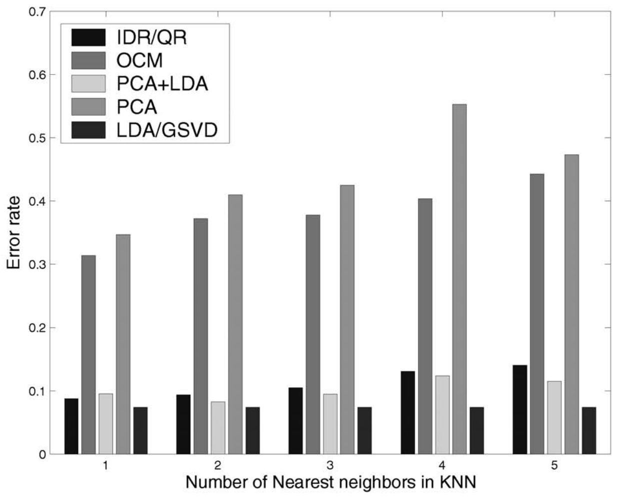
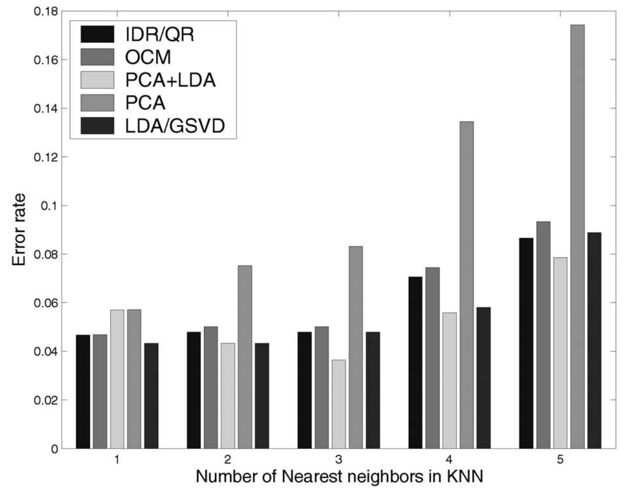


Fig. 5. Comparison of classification error rate on image data sets: ORL (left) and AR (right).



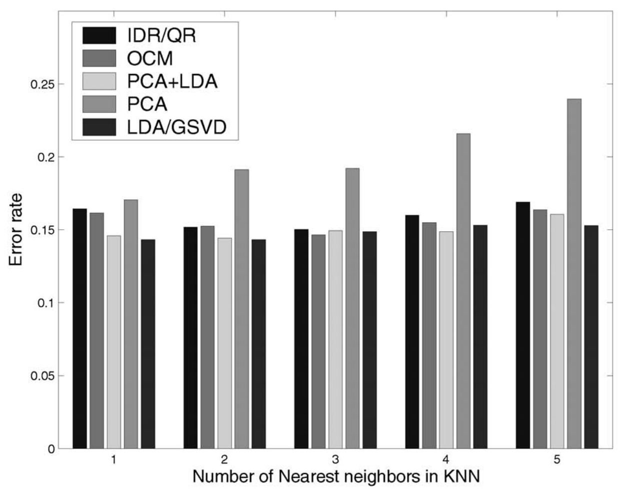


Fig. 6. Comparison of classification error rate on text document data sets: tr41 (left) and re0 (right).

6.3 The Performance of Incremental IDR/QR

In this experiment, we compare the performance of incremental IDR/QR with that of batch IDR/QR in terms of classification error rate and the computational cost. We randomly order the data items in the data set and insert them into the training set one by one incrementally with the given order. The remaining data is used as the test set. Initially, we select the first 30 percent data items as the training set. Incremental updating is then performed with the remaining data items inserted one at a time.

Fig. 9 shows the achieved classification error rates by batch IDR/QR and incremental IDR/QR on four data sets. In the figure, the horizontal axis shows the portion of training data items and the vertical axis indicates the classification error rate (as a percentage). We observe a trend that the error rate decreases when more and more training data items are involved. Another observation is that the error rate by incremental IDR/QR is quite close to that by batch IDR/QR. Indeed, on four data sets, the maximal error rate deviation between incremental IDR/QR and batch IDR/QR is within 4 percent. We use the paired t-test and the p-values for the four data sets: AR, ORL, tr41, and re1 are 0.0067, 0.0014, 0.0054, and 0.0045, respectively. The results quantify the above claim on the significance levels of the differences.

Recall that incremental IDR/QR is carried through QR Decomposition in three steps:

1. QR-updating of the centroid matrix C,
2. updating of the reduced within-class scatter W, and
3. updating of the reduced between-class scatter B.

The first and third steps are based on the exact scheme, while the second step involves approximation. Note that the main rationale behind our approximation scheme in updating W is that the change of Q matrix is relatively small and can be neglected for each single updating, where $C = QR$ is the QR Decomposition of C.

To give a concrete idea of the benefit of using incremental IDR/QR from the perspective of efficiency, we give a comparison of the compuational cost between batch IDR/QR and incremental IDR/QR. The experimental results are given in Fig. 10. As can be seen, the execution time of incremental IDR/QR is significantly smaller than

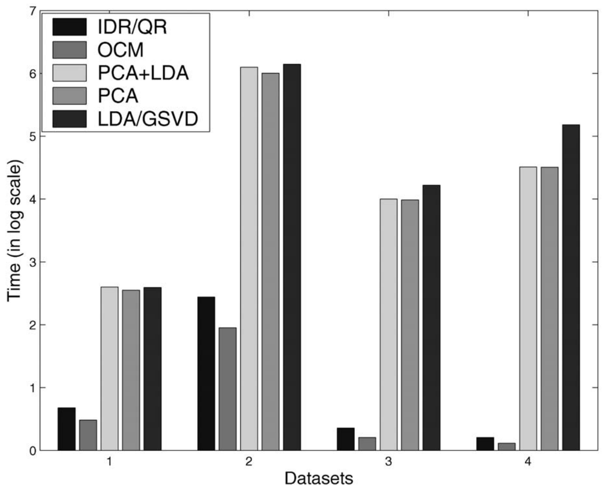


Fig. 7. Comparing the efficiency of computing the transformation (measured in seconds in log-scale). The horizontal axis denotes the four data sets: Data sets 1-4 corresponds to ORL, AR, tr41, and re0, respectively.

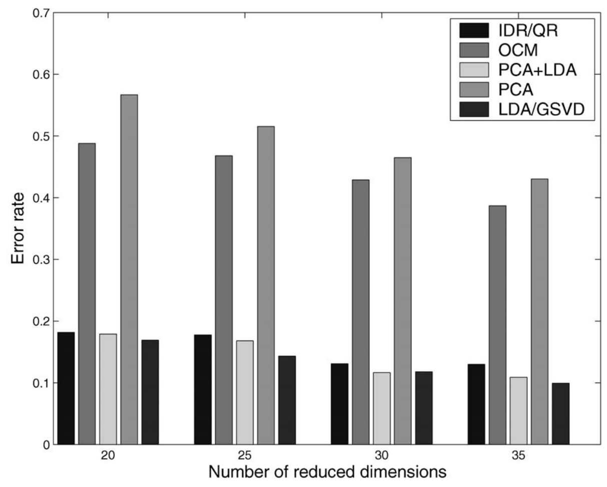


Fig. 8. The effect of small reduced dimension on classification error rate using AR.

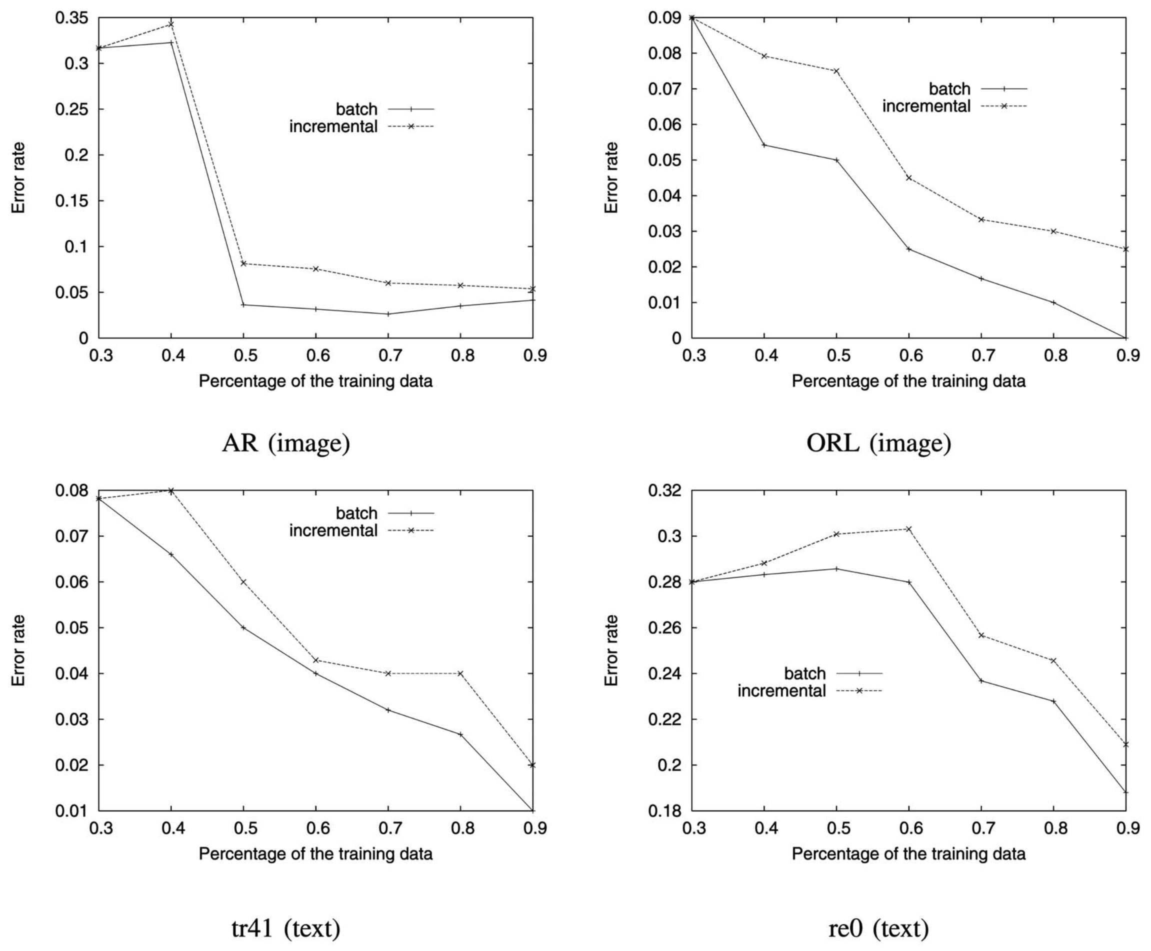


Fig. 9. Comparison of classification error rate between incremental IDR/QR and batch IDR/QR.

that of batch IDR/QR. Indeed, for a single updating, incremental IDR/QR takes $O(dk + k^3)$, while batch IDR/QR takes $O(ndk)$, where k is the number of classes in the current training set and n is the size of the current training set. The time for a single updating in incremental IDR/QR is almost a constant $O(dc + c^3)$, when all classes appear in the current training set and the speed-up of incremental IDR/QR over batch IDR/QR keeps increasing when more points are inserted into the training set. Note that we only count the time for Lines 1-6 in **Algorithm 1** in Fig. 1 since each updating in incremental IDR/QR only involves the updating of the QR Decomposition (Line 2), W (Line 5), and B (Line 6).

7 Conclusions

In this paper, we have proposed an LDA-based incremental dimension reduction algorithm, called IDR/QR, which applies QR Decomposition rather than SVD. The IDR/QR algorithm does not require the whole data matrix in main memory. This is desirable for large data sets. More importantly, the IDR/QR algorithm can work incrementally. In other words, when new data items are dynamically inserted, the computational cost of the IDR/QR algorithm can be kept small by applying efficient QR-updating techniques. In addition, our theoretical analysis indicates that the computational complexity of the IDR/QR algorithm is linear in the number of the data items in the training data set as well as the number of classes and the number of dimensions. Finally, our experimental results show that the accuracy achieved by the IDR/QR algorithm is very close to the best possible accuracy achieved by other LDA-based algorithms. However, the IDR/QR algorithm can be an order of magnitude faster. When dealing with dynamic updating, the computational advantage of IDR/QR over SVD or GSVD-based LDA algorithms becomes more dramatic while still achieving the comparable accuracy.

As for future research, we plan to investigate the applications of the IDR/QR algorithm for searching extremely high-dimenional multimedia data, such as video.

Appendix

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite and $G \in \mathbb{R}^{n\times q}$ have orthogonal columns, where $q \le n$. The following inequality holds*

$$\mathrm{trace}(G^T A G) \le \mathrm{trace}(A).$$

Proof. Let $\tilde{G} \in \mathbb{R}^{n\times(n-q)}$ be the matrix such that $[G, \tilde{G}]$ is orthogonal. That is,

$$[G, \tilde{G}] \cdot [G, \tilde{G}]^T = GG^T + \tilde{G}\tilde{G}^T = I_n,$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

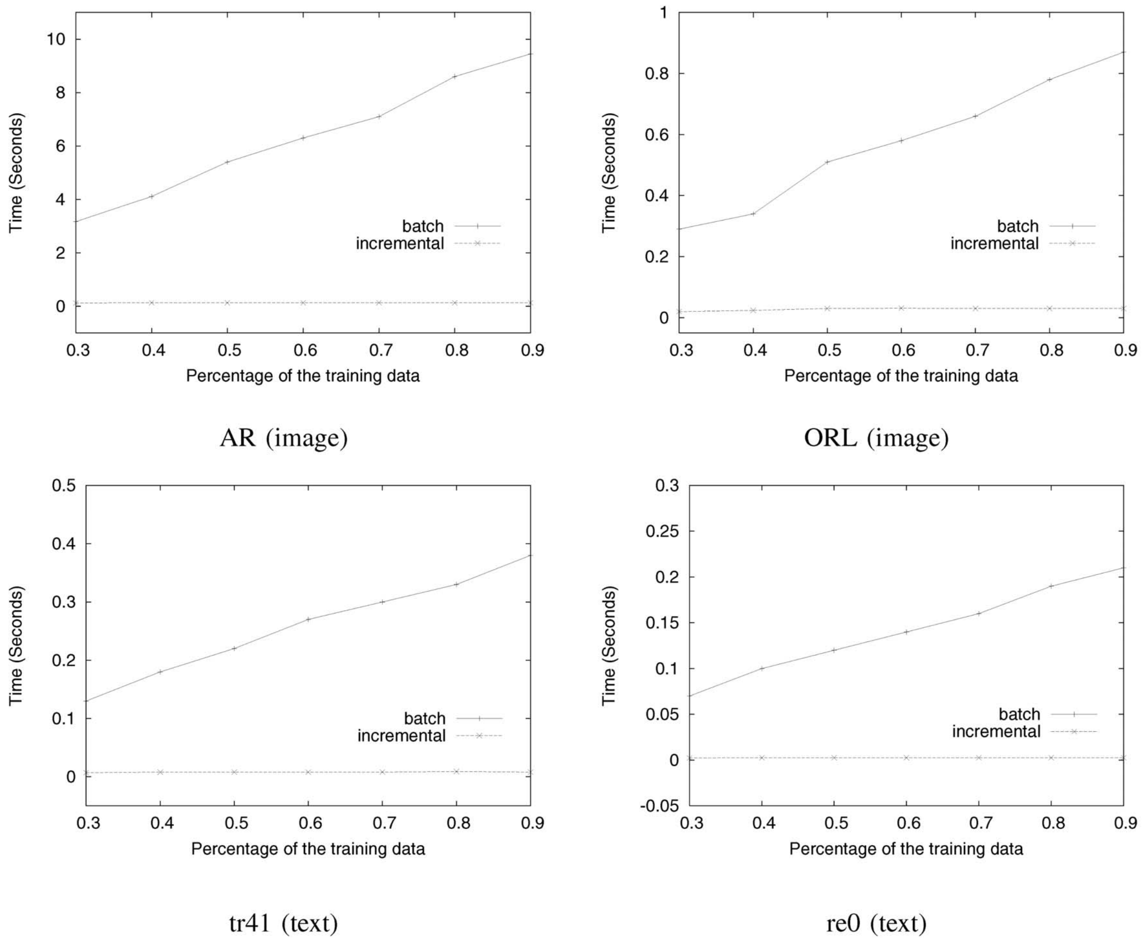


Fig. 10. Comparison of computional cost between incremental IDR/QR and batch IDR/QR.

It follows that

$$\begin{aligned}\mathrm{trace}(G^T A G) &= \mathrm{trace}(A G G^T) = \mathrm{trace}(A) - \mathrm{trace}(A \tilde{G} \tilde{G}^T) \\ &= \mathrm{trace}(A) - \mathrm{trace}(\tilde{G}^T A \tilde{G}) \le \mathrm{trace}(A),\end{aligned}$$

where the last inequality follows since $\tilde{G}^T A \tilde{G}$ is positive semidefinite. This completes the proof of the lemma. □

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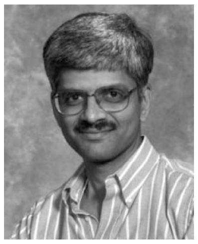

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